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Correction: The Zeros of the Partial Sums of the Exponential Function

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It has been pointed out to us by E. B. Saff that our proof [1], that there is a parabolic domain which is free of zeros of

$$S_n(z) = \sum_{k=0}^{n} z^k/k! \tag{1}$$

for n sufficiently large, is incorrect. We present here a proof that there is a parabolic domain (smaller than the one claimed in [1]) free of zeros of $S_n(z)$ for all n.

Write $z = x + iy$ and suppose $y^2 \leqslant cx$, where c is any positive number satisfying $ce^c < (\pi/2)$. For example $c = 0.7$ may be substituted in what follows.

Case (i). $0 \leqslant x \leqslant n$.

$$n!\, S_n(z)\, e^{-z} = \int_z^\infty t^n e^{-t}\, dt = \int_x^\infty t^n e^{-t}\, dt - \int_x^{x+iy} t^n e^{-t}\, dt, \tag{2}$$

so that

$$\begin{aligned} n!\, |S_n(z)| &\geqslant n!\, S_n(x) - \int_0^{|y|} |x \pm is|^n\, ds \\ &\geqslant n!\, S_n(x) - |y|\, (x^2 + y^2)^{n/2} \\ &\geqslant n!\, S_n(x) - |y|\, (x^2 + cx)^{n/2}. \end{aligned} \tag{3}$$

We claim next that $0 \leqslant x \leqslant n$ implies that

$$S_n(x) \geqslant \tfrac{1}{2}e^x. \tag{4}$$

To see this, note that in view of (2) it suffices to show that

$$\int_x^\infty s^n e^{-s}\, ds \geqslant \int_0^x s^n e^{-s}\, ds, \tag{5}$$

moreover, (5) holds for $0 \leqslant x \leqslant n$ if it holds for $x = n$. This is, in turn, a consequence of the inequality

$$(n + nu)^n\, e^{-(n+nu)} \geqslant (n - nu)^n\, e^{-(n-nu)}, \qquad 0 < u < 1,$$

or

$$(1 + u)\, e^{-(1+u)} \geqslant (1 - u)\, e^{-(1-u)}, \qquad 0 < u < 1,$$

or

$$(1 + u)/(1 - u) \geqslant e^{2u}, \qquad 0 < u < 1,$$

which is well known.

Using (4) in (3) we obtain

$$\begin{aligned} n!\, | S_n(z)| &\geqslant (n!e^x/2) - (nc)^{1/2}(x + (c/2))^n \\ &\geqslant (e^x/2)(n! - 2(nc)^{1/2}(x + (c/2))^n\, e^{-x}). \end{aligned}$$

But

$$(x + (c/2))^n\, e^{-x} \leqslant e^{(c/2)}(n/e)^n,$$

while $n! > (2\pi n)^{1/2}(n/e)^n$, and so

$$n!\, | S_n(z)| \geqslant (e^x/2)\, n^{1/2}((2\pi)^{1/2} - 2c^{1/2}e^{c/2})(n/e)^n > 0.$$

Case (ii). $n < x$. It is an easy consequence of the Eneström–Kakeya theorem on polynomials with monotone coefficients (see [2]) that all zeros of $S_n(z)$ lie in $| z | \leqslant n$, and so the region $x > n$ is free of zeros. This simple observation due to a student of Richard Varga, W. Ni, replaces an elaborate discussion of this case that we had devised.

Thus, we have shown that if $y^2 \leqslant cx$, $S_n(x + iy) \neq 0$ for any n.

References

1. D. J. Newman and T. J. Rivlin, The zeros of partial sums of the exponential function, *J. Approximation Theory*, **5** (1972), 405–412.
2. G. Pólya and G. Szegö, "Aufgaben und Lehrsätze," Vol. 1, Abschn. III, No. 23, Springer–Verlag, Berlin, 1954.